Calculate the limit

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$\lim_{n\to\infty}\left(\sqrt[3n+3]{(2n+1)!!} - \sqrt[3n]{(2n-1)!!}\right)\sqrt[3]{n^2}$.

Solution by Arkady Alt , **San Jose**, **California**, **USA**.

First note that $\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n} = \frac{2}{e}$. Indeed, since $\frac{\sqrt[n]{(2n-1)!!}}{n} = \sqrt[n]{a_n}$, where $a_n := \frac{(2n-1)!!}{n^n}, n \in \mathbb{N}$ and $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{(2n+1)!!}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n-1)!!}\right) =$
$\lim_{n\to\infty}\frac{2n+1}{n+1} \cdot \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{2}{e}$ then by Multiplicative Stolz-Cezaro Theorem
$\lim_{n\to\infty}\sqrt[n]{a_n} = \frac{2}{e}$.

Using that we obtain $\lim_{n\to\infty}\left(\sqrt[3n+3]{(2n+1)!!} - \sqrt[3n]{(2n-1)!!}\right)\sqrt[3]{n^2} = \lim_{n\to\infty}(\sqrt[n]{a_n})^{1/3}\left(\sqrt[3]{b_n} - 1\right)n =$
$\sqrt[3]{\frac{2}{e}}\lim_{n\to\infty} n\left(\sqrt[3]{b_n} - 1\right)$, where $b_n := \frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}$.

Since $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{\sqrt[n+1]{(2n+1)!!}}{n+1}\left(\frac{\sqrt[n]{(2n-1)!!}}{n}\right)^{-1} \cdot \frac{n+1}{n} = \frac{2}{e} \cdot \left(\frac{2}{e}\right)^{-1} \cdot 1 = 1$

then $\lim_{n\to\infty}\frac{\sqrt[3]{b_n} - 1}{\ln\left(\sqrt[3]{b_n}\right)} = 1$ and, therefore, $\lim_{n\to\infty} n\left(\sqrt[3]{b_n} - 1\right) = \lim_{n\to\infty} n\ln\left(\sqrt[3]{b_n}\right) =$
$\frac{1}{3}\lim_{n\to\infty} n\ln b_n = \frac{1}{3}\ln\left(\lim_{n\to\infty} b_n^n\right)$. Thus, to complete the solution remains to find $\lim_{n\to\infty} b_n^n$.

Since $b_n^n = \frac{2n+1}{\sqrt[n]{(2n+1)!!}}$ then $\lim_{n\to\infty} b_n^n = \lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n]{(2n+1)!!}} \cdot \frac{2n+1}{n+1}\right) = e$ and,

therefore, $\ln\left(\lim_{n\to\infty} b_n^n\right) = 1$. Hence, $\lim_{n\to\infty}\left(\sqrt[3n+3]{(2n+1)!!} - \sqrt[3n]{(2n-1)!!}\right)\sqrt[3]{n^2} = \frac{1}{3}\sqrt[3]{\frac{2}{e}}$.